NuMBerS numerical methods for biosciences students

Basic numerical skills: SHAPES AND SIZES – THE GEOMETRY OF TWO- AND THREE-DIMENSIONAL OBJECTS

Introduction

This helpsheet introduces the geometry of two- and three-dimensional objects. It is accompanied by a basic 'recipe sheet' that provides the formulae for perimeters, areas and volumes, and these can also be accessed in a spreadsheet that allows you to do these calculations online. The main aim of this helpsheet is not so much to provide the ways to carry out calculations, but to explore how these are used to explore real problems.

Two-dimensional objects (easy)

We are familiar with a range of shapes that exist in a single plane and are defined by one or more lines (sides) that together enclose the object. Regular objects are those that belong to a small number of families with simple properties, such as all sides are a straight line. A circle and related shapes called ellipses are defined by a single line that has no ends (that is it is joined back onto itself). Shapes can be defined by two lines, whose ends join and at least one of which must be curved – a semicircle (half circle) is a shape made up from one straight line and one curved line.

Shapes defined by straight lines alone must have at least three sides. Triangles have three sides, quadrilaterals have four sides, and shapes with more than four sides are collectively known as polygons. Within these classes, there may be special cases. For instance, a square is a quadrilateral where all sides are the same length, opposite pairs of sides are parallel to each other and each angle where adjacent sides meet is a right angle (90º). An equilaterial triangle has all three sides the same length, and each angle where adjacent sides meet is 60º.

Symmetry in two-dimensional objects (easy)

For many regular objects, you can draw one or more lines across them that divide them exactly in half in such a way that the two parts are mirror images of each other. This property is termed symmetry, and the line is described as an axis of symmetry. If a flat mirror is held along the axis of symmetry and vertical to the plane of the object, the half of the object that is visible and its reflection in the mirror are together identical in appearance to the original object. Examples of axes of symmetry are shown in Figure 1:

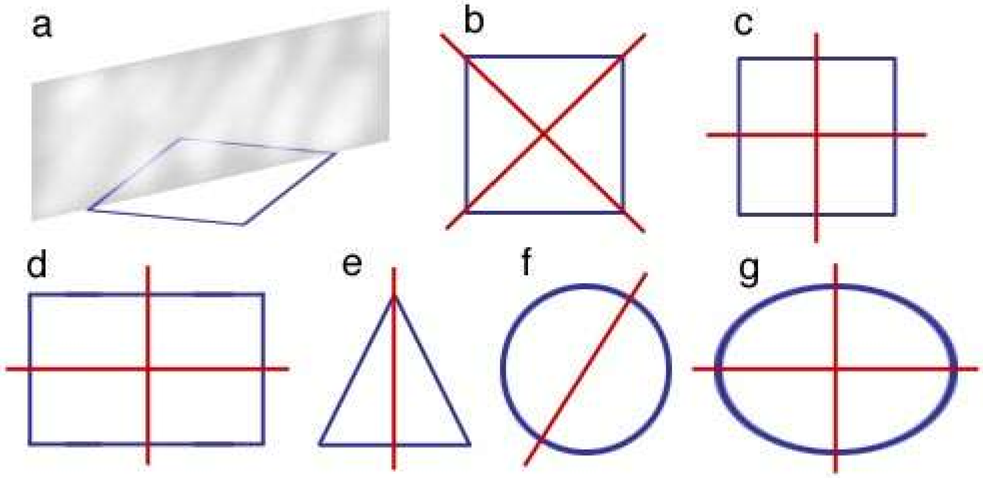

Figure 1. Symmetry in two dimensions. a: illustration of the principle of symmetry – a plane mirror is aligned vertically along the diagonal of a square, so that the visible half of the square and its reflection make a complete square; b: the two diagonal axes of symmetry for a square; c: two axes of symmetry that bisect opposite sides in a square; d: the two axes of symmetry in a rectangle, that bisect opposite sides; e: single axis of symmetry for an isosceles triangle (two equal sides) that bisects the base; f: any line passing through the centre of a circle is an axis of symmetry; g: an ellipsoid has two axes of symmetry, along the 'major' and 'minor' axes.

Three-dimensional objects (easy)

Regular three-dimensional (3-D) objects are commonly analogues of two-dimensional objects. Instead of being delimited by one-dimensional lines, they are defined by flat or curved planes. In the case of spheres (3-D analogue of a circle) and ellipsoids (3-D analogues of ellipses), a single and continuous curved plane delimits the object. Other 3-D objects have edges where planes meet. The family of cuboids are delimited by six planes, opposite pairs of which are parallel, and have straight edges.

A variety of 3-D objects collectively known as prisms have identically-shaped planes at their ends and the long sides made up of one or more planes forming a parallel-sided 'tube'. A cylinder is a prism with circular ends.

There is a corresponding family of tapered objects, where the base is a flat plane but the long sides are not parallel and the 'top' is a point. Such an object with a circular base is a cone, whilst a pyramid can have a triangular or quadrilateral base. A pyramid made up from four equilateral triangles is called a tetrahedron.

3-D objects delimited completely by polygons – polyhedra - are more complex. Many natural and man-made structures can be represented in this way – examples include virus capsids, footballs and geodesic domes.

As with 2-D objects, the area of 3-D objects is related to the square of linear dimension(s). Volume is a measure of the space that the object occupies in three dimensions, and is a function of the cube of the linear dimension(s).

Symmetry in three-dimensional objects (easy)

Just as the symmetry of 2-D objects is defined with respect to one or more lines (axes), the symmetry of 3-D objects can be defined with respect to one or more planes. Imagine the object being sliced into two equal pieces through a plane of symmetry. If the cut face is then placed against a flat mirror, the half of the object and its reflection in the mirror look identical to the original:

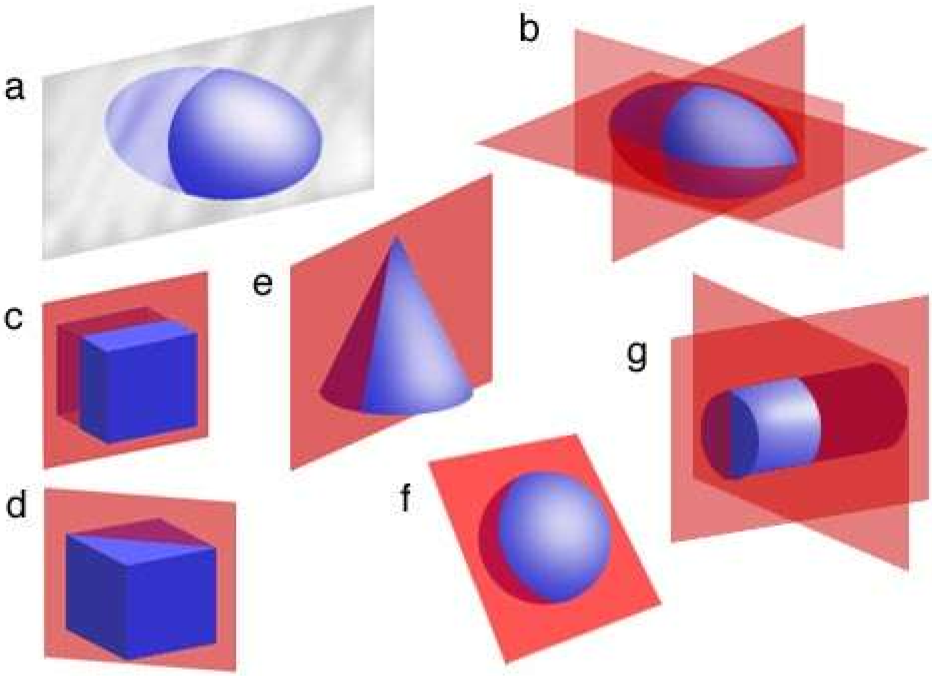

Figure 2. Symmetry in three dimensions. a: illustration of the principle of symmetry – an ellipsoid has been cut into two halves and the cut face of one half placed against a flat mirror to produce a reflection that recreates the original 3-D object; b: the three planes of symmetry for a triaxial ellipsoid; c and d: two of the nine planes of symmetry for a cuboid; e: any plane of symmetry for a cone passes through the vertex (point) and bisects the base; f: any plane that passes through the centre of a sphere is a plane of symmetry; g: a cylinder has a single plane of symmetry cutting it into two halves across the long axis, and any plane that bisects it lengthways.

Size-related properties (intermediate)

Several important properties of objects are determined by their size. Consider a mother seal and her pup living on the sea ice in Antarctica. The animals are homoiothermic ('warm-blooded') and they maintain their internal temperature by excess heat production by body tissue. Since the total heat production is a function of body mass, it will be related to volume, that is to the cube of body size (eg length). Heat loss, on the other hand, occurs through the surface, so is related to surface area or the square of body size. So if all other properties of mother and pup are the same, the balance between heat production and heat loss will be determined by the ratio between surface area and body volume. So if the mother is twice the size of the pup, we would expect her to lose heat at half the rate that the pup does.

The seal example is a very simple generalization, but it does represent an important concept in a variety of situations. An African elephant is the largest living land mammal. As with the seals, the ratio of its surface area to volume determines how it exchanges internal heat with the environment. Large animals have small surface area:volume (SA:V) ratios, and in the case of elephants living in the tropics their problem is more usually one of trying not to overheat, so they are almost hairless, carry large heat radiators (their ears) and use various behaviours to promote cooling and avoid overheating. Other size-related properties affect elephants. Whilst their bulk is determined by their volume (size cubed), the compressive strength of the legs that carry this bulk around is proportional to their cross-sectional area (size squared). So large land animals have proportionally thicker legs than small animals, and this in turn determines other features of their body plan and biomechanics (see Figure 3).

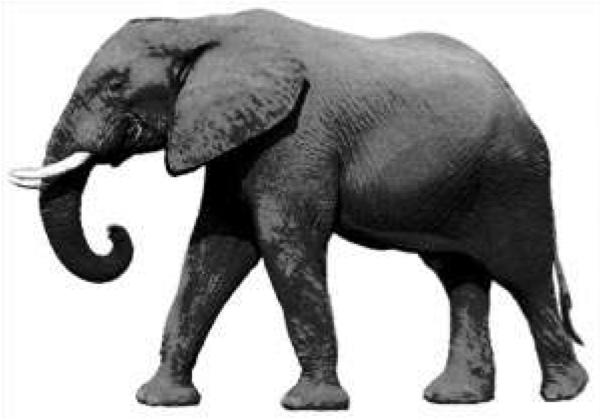

Figure 3. The African elephant is the largest living land animal, and several features of its biology are linked to its size and SA:V ratio.

Objects in fluids – the effects of size (advanced)

Observing micro-organisms moving through fluids, it is striking that as soon as they stop swimming they come to an abrupt halt. Contrast this with a supertanker that can take many kilometres to come to a stop. Both the microbes and the ship are moving through water that has very similar properties, so why do they behave differently?

Size is clearly important. On the one hand it determines the momentum of the body – the tendency to keep moving – and this is related directly to volume (size cubed). However, friction between the surface of the object and the water is a property of surface area (size squared). For different-sized bodies of similar density moving through the same fluid, the ratio between surface area and volume determines the ratio between friction (drag) and momentum.

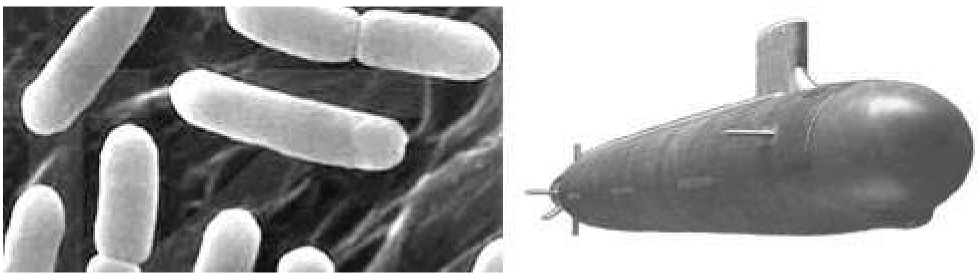

Figure 4. The rod-shaped bacteria are very similar in shape to the submarine, and both exist in similar fluid media. However, size determines how the two particles interact dynamically with their environment.

In the following table, we have calculated the SA:V ratio for a cylinder with hemispherical ends – the shape of both the bacteria and the submarine illustrated in Figure 4. You can see that a 1 µm bacteria has an SA:V ratio that is a hundred million times that for the 100 m submarine. Note that the huge values of SA:V for small objects is because the calculations are being performed with the same base unit (metre) throughout – however the same size-related differences would have been apparent if everything had been calculated in millimetres or micrometres.

Overall length	**Overall length (m)**	**Width (m)**	**Surface area (m^2)**	**Volume (m^3)**	**SA:V (m^{-1})**
1 µm	1×10^{-6}	5×10^{-8}	3.46×10^{-13}	8.25×10^{-21}	4.19×10^{7}
10 µm	1×10^{-5}	5×10^{-7}	3.46×10^{-11}	8.25×10^{-18}	4.19×10^{6}
100 µm	1×10^{-4}	5×10^{-6}	3.46×10^{-9}	8.25×10^{-15}	4.19×10^{5}
1 mm	1×10^{-3}	5×10^{-5}	3.46×10^{-7}	8.25×10^{-12}	4.19×10^{4}
10 mm	1×10^{-2}	5×10^{-4}	3.46×10^{-5}	8.25×10^{-9}	4.19×10^{3}
100 mm	1×10^{-1}	5×10^{-3}	3.46×10^{-3}	8.25×10^{-6}	4.19×10^{2}
1 m	1×10^{0}	5×10^{-2}	3.46×10^{-1}	8.25×10^{-3}	4.19×10^{1}
10 m	1×10^{1}	5×10^{-1}	3.46×10^{1}	8.25×10^{0}	4.19×10^{0}
100 m	1×10^{2}	5×10^{0}	3.46×10^{3}	8.25×10^{3}	4.19×10^{-1}

Note: If you are not familiar with the 'scientific' notation (eg 3.46×10^{-11}) used in this table, refer to the helpsheet 'Powers and logarithms'

The relationships between surface area and volume can be explored in the spreadsheet associated with this helpsheet.