NuMBerS
numerical methods for biosciences students

Basic numerical skills: POWERS AND LOGARITHMS

1. Introduction (easy)

Powers and logarithms provide a powerful way of representing large and small quantities, and performing complex calculations. Understanding powers will allow you to make better 'back-of-the-envelope' calculations or to quality-check results from your calculator.

Look at the number sequence:

10
100
1 000
10 000
100 000
1 000 000

Each of these numbers is the previous number multiplied by 10
This list can be re-written as (column 2):

10	=	10	1
100	=	10 x 10	2
1 000	=	10 x 10 x 10	3
10 000	=	10 x 10 x 10 x 10	4
100 000	=	10 x 10 x 10 x 10 x 10	5
1 000 000	=	10 x 10 x 10 x 10 x 10 x 10	6

The numbers in the right-hand column are the number of multiples of 10 for each number.
This is the POWER of 10 and is added as a superscript to 10 to represent each number.

So we can rewrite:

10	=	10^1
100	=	10^2
1 000	=	10^3
10 000	=	10^4
100 000	=	10^5
1 000 000	=	10^6

HINT: Count the zeros after 1 – this equals the power

This pattern continues below 10:

1	=	10 ÷ 10	=	10^0
0.1	=	10 ÷ 10 ÷ 10	=	10^{-1}

0.01	=	10 ÷ 10 ÷ 10 ÷ 10	=	10^{-2}
0.001	=	10 ÷ 10 ÷ 10 ÷ 10 ÷ 10	=	10^{-3}
0.0001	=	10 ÷ 10 ÷ 10 ÷ 10 ÷ 10 ÷ 10	=	10^{-4}

HINT: Count the number of zeros after the decimal place and before the first 'one', then add one for the decimal point – this equals the 'minus' power

Note that minus power indicates a value that is the reciprocal of the positive power, thus:

$$10^{-3} = 0.001 = 1 \div 1000 = 1 \div 10^{3}$$

2. Powers and bases (intermediate)

In the previous section, we expressed numbers as powers of 10. This is termed the **base** of the expression. However the base can be any number so that:

100	=	10 x 10	=	10^2	Base 10:Power 2
16	=	2 x 2 x 2 x 2	=	2^4	Base 2 :Power 4
27	=	3 x 3 x 3	=	3^3	Base 3 :Power 3
625	=	5 x 5 x 5 x 5	=	5^4	Base 5 :Power 4

and so on …

In **power notation** (also called **scientific notation**) any number can be represented as a power of 10 so that:

325 is between 100 (10^2) and 1,000 (10^3) and is equal to 3.25 x 100 = 3.25 x 10^2

625 400 is between 10^5 and 10^6 and is equal to 6.254 x 10^5

There are two **special cases** of powers. Any number raised to the power of 1 is itself, and any number raised to the power 0 equals 1.

3. Multiplying and dividing using powers (intermediate)

You can perform fairly daunting calculations by using a simple property of powers. First, look at a very simple sum like:

100 x 1 000 000

It's obvious that the answer is 100 000 000 (one hundred million), but now look at this when we write the numbers as powers:

$$10^2 \times 10^6 = 10^8 = 10^{(6+2)}$$

You will have spotted that the superscript for the answer (8) is the sum of the two superscripts of the multipliers (2 and 6). Thus we can multiply quantities expressed as exact powers of the same base by adding their exponents. Similarly, if we want to divide, we subtract the powers:

$$1\,000\,000 / 100 = 10^6 / 10^2 = 10^{(6-2)} = 10^4 = 10\,000$$

Easy with exact multiples of ten, but how is this used with 'real' numbers? You have already seen that you can express a number as a multiplier and a power of ten (commonly termed '**scientific notation**'). So that:

$$3756 = 3.756 \times 10^3 \text{ and } 696 = 6.96 \times 10^2$$

Using the same rule as we applied to exact powers of ten, we can multiply these two numbers:

$$3756 \times 696 = 3.756 \times 10^3 \times 6.96 \times 10^2$$

Again, we can add the powers of ten to make one multiplier:

$$3.756 \times 10^3 \times 6.96 \times 10^2 = 3.756 \times 6.96 \times 10^5$$

Then we multiply the other two terms:

$$3.756 \times 6.96 \times 10^5 = 26.204 \times 10^5 (= 2.6204 \times 10^6)$$

Perhaps not all that exciting in itself, but it does help with doing mental arithmetic approximations. Taking the same sum and rounding up the multipliers:

$$3.75 \times 7 = 21 + 5 = 26$$

$$\text{and } 10^3 \times 10^2 = 10^5,$$

$$\text{then: } 3756 \times 696 \sim 26 \times 10^5$$

4. 'Powers of powers' (intermediate)

In the last section, we multiplied powers of ten (or numbers expressed as a figure multiplied by a power of ten) by adding the powers together. How do we deal with a situation where we want to evaluate an expression where a number expressed as a power of ten is itself raised to a power? For instance, what is the square of one billion?

$$1\,000\,000\,000 \times 1\,000\,000\,000 = 1\,000\,000\,000\,000\,000\,000$$

Written in power notation, this becomes:

$$(10^9)^2 = 10^{18}$$

Notice that the superscript of the answer is the **product** (not the sum) of the superscripts of the expression to the left of the equals sign. Similarly, we can raise to a power a number expressed in scientific notation:

$$(5 \times 10^7)^3 = 5^3 \times (10^7)^3 = 5^3 \times 10^{21} = 1.25 \times 10^{22}$$

5. Roots (advanced)

So far we have looked at powers of ten that are whole number, either greater than one for numbers exceeding ten (eg $100 = 10^2$) or less than one to represent one divided by the power of ten (eg $1 \div 1000 = 0.001 = 10^{-3}$). We can also use power notation to represent roots. For instance the square root of one hundred is written as:

$$\sqrt{100} = 100^{½} \text{ or } 100^{0.5}$$

Using what you learnt about powers of powers in the previous section, you should be able to see that:

$$100^{0.5} = (10^2)^{0.5} = 10^{(2 \times 0.5)} = 10^1$$

The cube root raises the number to power one-third eg $1000^{0.333}$ and the fifth root raises the number to power one-fifth eg $1000^{0.2}$.

More generally, the 'xth' root of A is given by $A^{(1/x)}$.

6. Logarithms (advanced)

In section 3, we concentrated on writing numbers as a multiplier and a power of ten. Alternatively the numbers could be written with fractional powers of 10 as:
$325 = 10^{2.512}$, $625\ 400 = 10^{5.796}$, $0.00256 = 10^{-2.592}$

This is more complicated than power notation and requires a calculator or set of tables and is principally of use when dealing with logarithms:

Logarithms are the reverse of **powers**.

The logarithm is the power of a number to a particular base. Taking ten as the base, and writing the logarithm to base 10 as '$\log_{10}$':

$$\log_{10}(100) = \log_{10}(10^2) = 2$$
$$\log_{10}(1000) = \log_{10}(10^3) = 3$$
$$\log_{10}(0.01) = \log_{10}(10^{-2}) = -2$$

We saw above that an alternative way to write numbers is to use fractional powers of 10:

$$325 = 10^{2.512},\ 625\ 400 = 10^{5.796},\ \text{and } 0.00256 = 10^{-2.592}$$

We can therefore express these numbers as logarithms to base 10:

$$\log_{10}(325) = 2.512$$
$$\log_{10}(625\ 400) = 5.796$$
$$\log_{10}(0.00256) = -2.592$$

Why bother? Well logarithms are useful for a number of reasons, one of which is to do complex multiplications without a calculator. When multiplying two numbers you can arrive at an answer by looking up the $\log_{10}$ of the two numbers, adding the logs and then looking up the antilog$_{10}$ of the result. In today's world of calculators this technique is not used much, but again it is sometimes a quick way to estimate the result of awkward and complex calculations. Occasionally, you may use logarithms and antilogarithms when solving equations. For instance:

$$\text{If } 10^x = 625\ 400,$$

$$\text{then } x = \log_{10}(625\ 400) = 5.796$$

Similarly, you can work out a power of a number using logarithms. To work out the square of 5624, first find the $\log_{10}$ (3.75) and then multiply this by 2 (the power). The antilog_{10} of the result (7.5) is 3.162×10^7, which agrees with the answer that my calculator gives (3 1629 376). The reverse of this operation is to find the square root of the answer, which can be written as $(3.162 \times 10^7)^{0.5}$. Dividing the $\log_{10}$ (7.5) by two gives a result of 3.75, whose antilog_{10} equals 5624, which is where we started!

As with powers, logarithms can have different bases. A common base used for both powers and logarithms is the constant e. For an explanation of e see the separate helpsheet on exponential growth. Logarithms using the base e are called **natural logarithms** or **Naperian logarithms** and use the symbol ln (not $\log_e$)

Powers and logarithms using base e are very useful for calculation of population growth, which is covered in a separate resource on the **NuMBerS** site.

7. pH – applying logarithms in the real world (advanced)

In chemistry you will come across the units of **pH**. These are a measure of acidity and are a special case of logarithms. The letter **p** is used to denote the expression $-\log_{10}$. **H** denotes hydrogen ion concentration, or $[H^+]$, in mol l^{-1} so that pH actually means: $-\log_{10} [H^+]$.

pH runs from 0 to 14 so that a pH of 2 has a concentration of hydrogen ions of 10^{-2} mol l^{-1} which is very acidic, a pH of 7 has a concentration of hydrogen ions of 10^{-7} mol l^{-1} which is considered neutral whereas a pH of 13 has a concentration of hydrogen ions of 10^{-13} mol l^{-1} which is very alkaline.