NuMBerS
numerical methods for
biosciences students

Basic numerical skills: EQUATIONS AND HOW TO SOLVE THEM

1. Introduction (really easy)

An equation represents the equivalence between two quantities. The two sides of the equation are in balance, and solving equations involves maintaining the balance whilst rearranging different parts of the equation.

Equations are used to demonstrate relationships. A very simple example is:

$$2 + 5 = 7$$

So long as the same thing is done to both sides, the equation remains in balance. So if you subtract 2 from both sides:

$$2 + 5 - 2 = 7 - 2$$

$$5 + (2 - 2) = 7 - 2$$

$$5 = 5$$

Now replace one of the numbers with an unknown, represented by the symbol x:

$$x + 5 = 7$$

To find the value of x, you need to rearrange the equation so that the unknown (x) is on one side of the ‘equals’ sign, and everything else is on the opposite side. In this case, subtracting 5 from each side of the equation will leave the x on its own:

$$x + 5 - 5 = 7 - 5$$

$$x + 0 = 2$$

2. Adding and subtracting (easy)

In the simple examples in the previous section, we established that we could do anything to an equation, so long as we did the same thing to both sides. In adding or subtracting, we simply did the same operation to each side of the equation, for instance:

$$x - 7 = 13$$

$$x - 7 + 7 = 13 + 7$$

$$x + 0 = 20$$

Adding 7 to both sides of the equation is designed to cancel out the -7 on the left-hand side of the equation. A simple way to shortcut this step is simply to move the -7 to the right-hand side of the equation and to change its sign. Thus:

$$x - 7 = 13$$

$$x = 13 + 7$$

$$x = 20$$

Similarly with an addition:

$$3x + 8 = 26$$

$$3x = 26 - 8$$

$$3x = 18$$

3. Multiplication and division (easy)

The last equation in the previous section ended up as:

$$3x = 18$$

The expression $3x$ represents the value of x multiplied by three. The aim of the exercise is to find the value of x. Again, we need to transform both sides of the equation so that we are left with the unknown on one side of the equation. Here, we divide both sides by 3:

$$3x/3 = 18/3$$

$$x = 18/3 = 6$$

The short cut method is simply to move the multiplier to the other side of the equals sign, and turn it into a divisor, thus:

$$3x = 18$$

$$x = 18/3$$

$$x = 6$$

Similarly, a divisor becomes a multiplier when it is moved across to the other side of the equation:

$$x/7 = 5$$

$$x = 7 \times 5 = 35$$

4. Powers (intermediate)

An equation may contain a quantity raised to a power. Again, so long as the same operation is performed on both sides, the equation can be transformed to solve for an unknown variable. Taking a very simple example:

$$x^2 = 25$$

$$\sqrt{x^2} = \sqrt{25}$$

$$x = 5$$

The symbol √ indicates the square root of a number - note that the square root can also be written as $x^{0.5}$. You need to take care when dealing with other quantities in the equation, so that these are transformed first and leave the power as the final transformation. Thus:

$$3x^2 + 7 = 19$$

First shift the offset, +7:

$$3x^2 = 19 - 7$$

Then deal with the multiplier, 3:

$$x^2 = (19 - 7)/3 = 12/3 = 4$$

Finally, use square root of each side to solve for x,

$$x^2 = 4$$

$$\sqrt{x^2} = \sqrt{4}$$

$$x = 2$$

If you try to do these operations in a different order, you will soon get into a rather complex calculation where there is plenty of scope for making mistakes.

You can solve equations involving higher powers in the same way, provided that there is only one power in the equation or the equation can be transformed to yield a single power. However, if you are faced by an equation that contains more than one power you will need to use a formula to solve it. For instance, you cannot use the methods described here to solve this equation:

$$2x^2 + 3x = 29$$

5. Transforming equations (advanced)

Now that you understand the basic operations, try some more complex transformations that allow you to express one variable in terms of others. Starting with a simple equation linking speed (u), distance (s) and time (t), it is easy to see that, by using the rules you have already established:

$$\text{if} \quad u = s/t$$

$$\text{then} \quad s = ut$$

$$\text{and} \quad t = s/u$$

This is for an object traveling at a constant speed. If the object is accelerating at a constant rate, its speed increases with time. If the initial speed is still denoted by u and its speed at a time t is represented by v, then the constant acceleration (a) is given by:

$$a = (v - u)/t$$

If you know the acceleration, you can transform this equation to find the value of v at a particular time:

$$a = (v - u)/t$$

$$at = v - u$$

$$v = u + at$$

From this relationship, you can find the distance traveled (s) in a given time (t). At the start of this section, you saw that distance is speed multiplied by time. When an object is under constant acceleration, the average speed at a time t is half of the sum of the initial speed (u) and the speed (v) at time t. So you can calculate distance traveled as:

$$s = 0.5(u + v)t$$

If you don't know the value of v but do know the value of a, you can replace the value of v in the last equation:

$$v = u + at$$

$$s = 0.5(u + v)t$$

$$s = 0.5(u + u + at)t$$

$$s = ut + 0.5\,at^2$$

(Obviously, if the object is accelerating from rest, u is zero, and the equation becomes: $s = 0.5\,at^2$)

Alternatively, you can rearrange the same equation to find the value for v when you know u, a and s but not t. Time (t) can be expressed in terms of velocity and acceleration:

$$v = u + at$$

$$at = u - v$$

$$t = (v - u)/a$$

You can substitute this expression into the earlier equation:

$$s = ut + 0.5\,at^2$$

$$s = u\,(v - u)/a + 0.5\,a\,[(v - u)/a]^2$$

This looks quite daunting. It helps to work out the most complex expression separately. This is represented by a 'stand-in' variable, shown here as A:

$$s = u\,(v - u)/a + A$$

$$A = 0.5\,a\,[(v - u)/a]^2$$

$$A = 0.5\,a\,(v - u)^2/a^2$$

The expression $(v\text{-}u)^2$ means $(v\text{-}u)$ multiplied by $(v\text{-}u)$ so that A now becomes:

$$A = 0.5\,a\,(v^2 - 2uv + u^2)/a^2$$

$$A = (0.5v^2/\text{a}) - (uv/a) + (0.5u^2/a)$$

Having dealt with the complex expression, it can be plugged back into the equation:

$$s = u\,(v - u)/a + A$$

$$s = u\,(v - u)/a + 0.5v^2/\text{a} - uv/a + 0.5u^2/a$$

$$s = uv/a - u^2/a + 0.5v^2/\text{a} - uv/a + 0.5u^2/a$$

$$as = uv - u^2 + 0.5v^2 - uv + 0.5u^2$$

$$as = 0.5v^2 - 0.5u^2$$

$$v^2 = u^2 + 2as$$

This section has shown how the simple transforms presented in this document can be used to rearrange equations to derive new relationships between variables, to meet different needs. In this example, the main equations of motion under constant acceleration have been derived from simple expressions relating speed, distance and time. If you understood the much simpler equations earlier in this document, you should have been able to follow the different transformations in this section.

6. Simultaneous equations (advanced)

If you want to find out the value of an unknown, you need to know the value of everything else in the equation. So if you have an equation with two unknowns, like:

$$4x + 12y = 44$$

it is impossible to find unique solutions for x and y. However, if you have a second equation containing the same unknowns, you can use the same techniques as you have been using on single equations to find the values of both unknowns.

$$4x + 12y = 44$$

$$7x + 4y = 26$$

The technique used to find the values of the unknowns in these equations is quite simple. You need to change one or both of them so that the multiplier for one of the unknowns is the same in both equations. In this case it is easy to see that the multiplier for y in the second equation is exactly one-third of the value in the first. You know that you can keep the equation balanced if the same operation is performed on both sides. Here, we will multiply both sides by three:

$$3 \times (7x + 4y) = 3 \times 26$$

$$21x + 12y = 78$$

If we subtract one equation from the other, the y terms will cancel out, leaving only x terms. This will leave a simple equation in x.

$$21x + 12y = 78$$
$$4x + 12y = 44$$

subtraction leaves:

$$17x = 34$$
$$x = 2$$

The solution to x can then be added to either of the original equations to find y:

$$7x + 4y = 26$$

$$14 + 4y = 26$$

$$y = 3$$