Introduction to functions and models: INTRODUCING THE QUADRATIC FUNCTION

1. Introduction (easy)

Quadratic functions are equations containing a squared term. When plotted, a quadratic function describes a curve called a **parabola**, as in the flight of a projectile subject to gravity (see Section 3). Quadratic functions are often used to simulate quantities that either increase to a maximum value and then fall away, or decrease to a minimum value and then increase again.

2. The quadratic function explored (simple)

The simplest quadratic equation would look like:

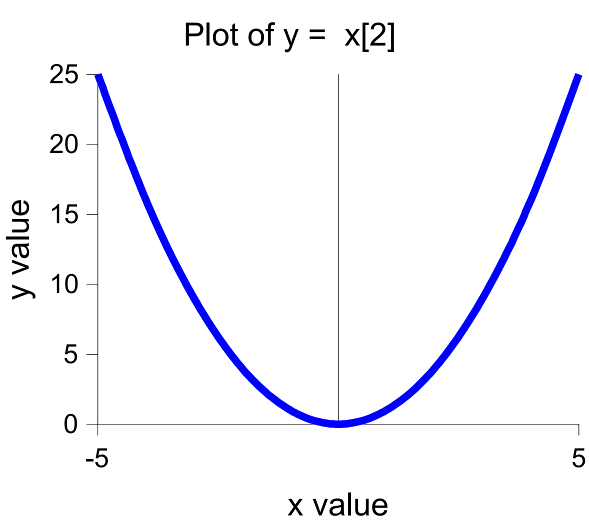
$$y = x^2$$

More generally, a quadratic equation will have the form:

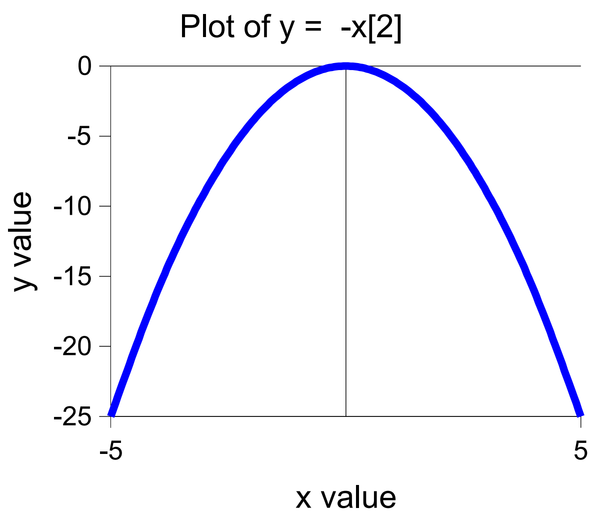
$$y = ax^2 + bx + c$$

where a, b and c are the **parameters** or **coefficients** of the equation. The parameters can be positive or negative, and a must be non-zero, whilst b and/or c can be zero.

The parabola described by the quadratic equation is a consequence of the squared term. Take the simple example at the start of this section, $y = x^2$. It is clear that when x = 0, y also equals zero. When x = 2, y = 4, but when x = -2 again y = 4. A plot of this function looks like this:



If the parameter a has a negative value, the parabola is inverted. For instance, the equation $y = -x^2$ looks like this:



Section 4 explores the effects of changing other parameters in the quadratic equation, to produce a variety of parabolas.

3. What goes up must come down - the flight of projectiles (intermediate)

When a projectile is fired vertically, it has an initial upwards velocity as it leaves the muzzle of the gun. The downwards attraction of gravity slows the projectile, until it reaches a height when the upwards velocity is zero. The projectile then falls back to earth, accelerating under gravity (assuming no air resistance).

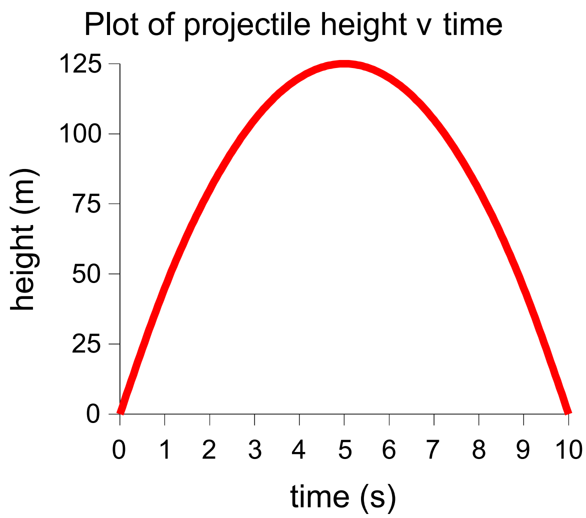
The equation of motion that relates distance travelled, time, initial velocity and acceleration is a quadratic:

$$s = ut + 0.5gt^2$$

where: s is the distance travelled at time t,
u is the initial velocity, and
g is the acceleration

Note that in the projectile example, the height and the initial velocity are both positive (ie upwards), but the acceleration due to gravity acts in the opposite direction and so has a negative value. So the coefficient for the squared term will be negative, giving the inverted parabola shown in the second example in Section 2.

If we put some values into the equation, g = -10 m s^{-2} and u = 50 m s^{-1}, we can then plot the height of the projectile over time:



Note that this plot has a horizontal axis of time, but it looks very much like the path that a projectile would take in space if it were fired at an angle to the horizontal rather than vertically. We can use the quadratic equation to construct a model of projectile flight. The first part is to work out how fast the projectile is travelling vertically and how fast it travels horizontally when fired from a gun at an angle θ to the horizontal.

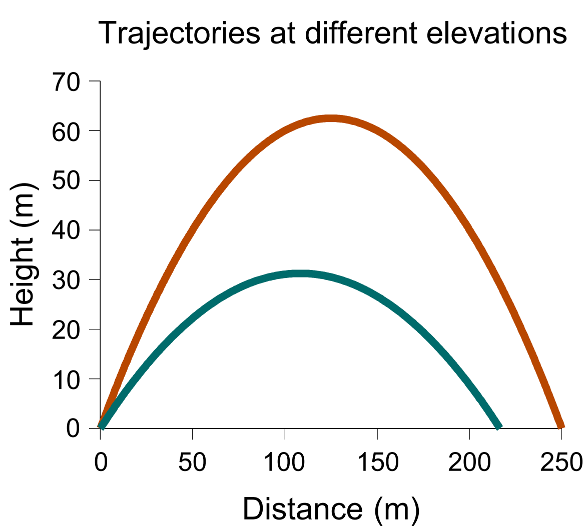
Initial vertical velocity is given by: $u_v = u \sin\theta$

Horizontal velocity is assumed to be constant: $u_h = u \cos\theta$

So at a given time t, the height of the projectile is: $s_v = u_v t + 0.5gt^2$

and the distance travelled horizontally is: $s_h = u_h t$

We can calculate a trajectory for any values of θ between 0° (horizontal) and 90° (vertical). Taking a muzzle velocity u = 50 m s^{-1}, and muzzle elevation angles $\theta = 30°$ and $\theta = 45°$ gives the following trajectories:



The muzzle elevation of θ = 45° gives a greater vertical component of velocity, so that the projectile not only travels to a greater height, but also travels further horizontally.

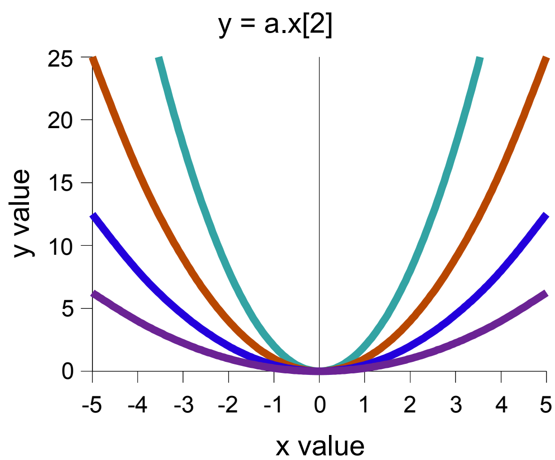
4. Exploring the parameters of the quadratic function (intermediate)

In Section 2, you saw that the basic quadratic equation has three parameters:

$$y = ax^2 + bx + c$$

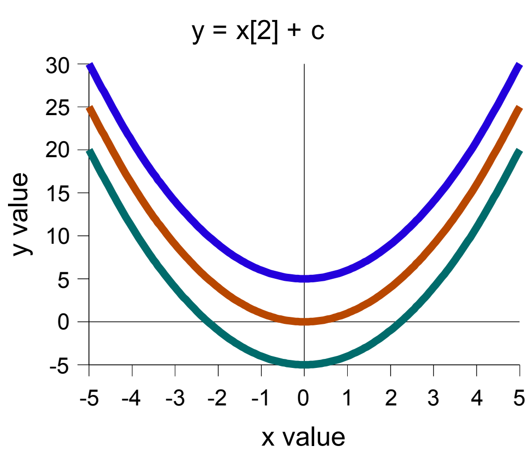
The parameter a has to be non-zero, and if it is positive the parabola has a minimum value and is open at the top, like a cup. If a is negative, the parabola has a maximum value and is open at the bottom, like a dome.

Increasing the value of a makes the parabola steeper, whilst decreasing a makes the parabola shallower. If we plot $y = x^2$ and $y = 0.25x^2$, $y = 0.5x^2$, and $y = 2x^2$, the results look like this:



The shallowest parabola is $y = 0.25x^2$, and the steepest is $y = 2x^2$.

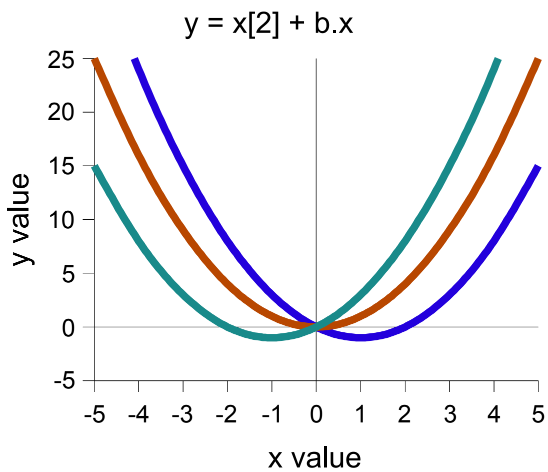
What effect does changing the other parameters have? The simplest one to understand is the constant, c, so we will keep b = 0 initially. For a given value of a, this simply acts as a vertical offset. If we plot $y = x^2$, $y = x^2 + 5$ and $y = x^2 - 5$, the results look like this:



The effect of changing the value of c is the same for an inverted parabola (when a is negative).

If we turn our attention to b, the coefficient for the x-term, we see that the curve changes location. If $c = 0$, a positive value of b moves the parabola down and to the right, so that the minimum occurs at a higher value of x but a lower value of y. A negative value of b shifts the parabola down and to the left, so that the minimum occurs at a lower value of x and a lower value of y.

The following plots are for $y = x^2 - 2x$, $y = x^2$ and $y = x^2 + 2x$:



In the case of an inverted parabola (a is negative), a positive value of b shifts the parabola up and to the right, that is the maximum has higher x- and y- values. A negative value of b shifts the parabola up and to the left, so that the maximum has a lower x-value but a higher y-value.

The following plots are for $y = -x^2 - 2x$, $y = -x^2$ and $y = -x^2 + 2x$: