NuMBerS numerical methods for biosciences students

Basic numerical skills: FRACTIONS, DECIMALS, PROPORTIONS, RATIOS AND PERCENTAGES

1. Introduction (simple)

This helpsheet is concerned with the ways that we express quantities that are not whole numbers, and how we express numbers in relation to other numbers. You may also need to refer to helpsheets on powers, measurements, and concentrations and dilutions.

All of the topics that are covered here will be familiar from everyday life, but you may have already found problems with the calculations or with converting from one form to another.

2. Fractions, and adding and subtracting fractions (simple)

A fraction is simply one number divided by another, for instance ¾ is 3 (called the numerator) divided by 4 (called the denominator).

$$\frac{3}{4} = 3 \div 4$$

If the numerator is equal to the denominator, the fraction has the exact value of one. If the numerator is larger than the denominator, the fraction is termed an improper fraction, and it can be expressed as a whole number and a proper fraction. For instance:

$$\frac{11}{4} = 2\frac{3}{4}$$

where 4 divided into 11 gives 2, leaving a remainder of 3. Similarly, if we have a mixed number, we can express this as an improper fraction:

$$5\frac{1}{4} = \frac{21}{4}$$

where 5 is multiplied by 4 (the denominator) to give 20, and then the extra ¼ is added to give 21. The logic of this process will (should, may) become clearer when we look at adding fractions together.

2.1 Adding fractions (fairly simple)

Adding fractions together is simplest if the denominator is identical. For instance:

$$\frac{5}{3} + \frac{8}{3} = \frac{(5+8)}{3} = \frac{13}{3} = 4\frac{1}{3}$$

Note that the denominator remains unchanged in this process. What we did was to add five objects, each with a value of one third, to eight objects with the same individual value.

What happens if the denominators in the two fractions are not identical? One or both fractions have to be transformed so that they have a common (shared) denominator. In this example, we have expressed the mixed number used in the example above as two separate fractions, by writing the whole number as a fraction whose denominator is 1:

$$5\frac{1}{4}=\frac{5}{1}+\frac{1}{4}$$

Here, we can convert both denominators to 4:

$$\frac{5}{1}+\frac{1}{4}=\frac{(5\times4)}{(1\times4)}+\frac{1}{4}=\frac{20}{4}+\frac{1}{4}=\frac{(20+1)}{4}=\frac{21}{4}$$

Note that both numerator and denominator are multiplied by the same amount. So long as you carry out exactly the same multiplication to top and bottom, the value of a fraction remains unchanged:

$$\frac{15}{3}=\frac{30}{6}=\frac{(7\times30)}{(7\times6)}$$

In the example, the denominator and numerator have both been increased. If it is possible to decrease the denominator, the fraction is said to have been simplified:

$$\frac{210}{42}=\frac{(210\div14)}{(42\div14)}=\frac{15}{3}=\frac{5}{1}=5$$

If when adding two fractions there is no simple way to make the denominators equal (as is the case when one is an exact multiple of the other), the denominators are multiplied together to make a new common denominator before the fractions are added:

$$\frac{12}{6}+\frac{11}{7}=\frac{(12\times7)}{(6\times7)}+\frac{(11\times6)}{(7\times6)}=\frac{(84+66)}{42}=\frac{158}{42}$$

The improper fraction in the answer can be converted to a mixed number, and it may then be possible to simplify the fraction part:

$$\frac{158}{42}=3\frac{24}{42}=3\frac{(24\div6)}{(42\div6)}=3\frac{4}{7}$$

2.2 Subtracting fractions (still fairly simple)

The methods used to subtract fractions are the same as are used to add them together. If the denominator is the same for both fractions:

$$\frac{5}{8}-\frac{3}{8}=\frac{(5-3)}{8}=\frac{2}{8}=\frac{1}{4}$$

If the two fractions do not have an obvious common denominator, the two denominators are multiplied together:

$$\frac{12}{6}-\frac{11}{7}=\frac{(12\times7)}{(6\times7)}-\frac{(11\times6)}{(7\times6)}=\frac{(84-66)}{42}=\frac{18}{42}=\frac{3}{7}$$

This is a slightly contrived example designed to yield an answer that simplifies easily. In this case it would have been simpler to have started by converting both fractions to mixed numbers:

$$\frac{12}{6}-\frac{11}{7}=2-1\,\frac{4}{7}=1-\frac{4}{7}=\frac{3}{7}$$

The choice of whether to work with improper fractions or mixed numbers will be based on the sizes of numerators and denominators.

2.3 More complex additions and subtractions (a bit more complicated)

Complex sums with more than two fractions can be carried out, but it is advisable to break these down into smaller units:

$$\frac{13}{8}-\frac{2}{7}+\frac{20}{3}-\frac{9}{5}$$

has the common denominator 840! It is easier to rearrange into two additions and then do a subtraction with the results - this also sorts out the mixture of signs in the expression:

$$\left(\frac{13}{8}+\frac{20}{3}\right)-\left(\frac{2}{7}+\frac{9}{5}\right)$$

Notice that the first set of brackets contains the sum of the positive fractions, whilst the second set is the sum of the negative fractions in the original expression. These can be evaluated:

$$\frac{[(13\times3)+(20\times8)]}{24}-\frac{[(2\times5)+(9\times7)]}{35}=\frac{199}{24}-\frac{78}{35}$$

$$\frac{[(199\times35)-(78\times24)]}{(24\times35)}=\frac{5093}{840}=5\,\frac{53}{840}$$

Brackets were used to break the expression into separate units and to address ambiguities that could have arisen from the mixture of signs in the expression. Note that the expressions moved within brackets may change sign, for instance:

$\frac{4}{5}-\frac{5}{7}-\frac{3}{4}$ can also be written as $\frac{4}{5}-\left(\frac{5}{7}+\frac{3}{4}\right)$, but is not the same as $\frac{4}{5}-\left(\frac{5}{7}-\frac{3}{4}\right)$

3. Multiplying and dividing fractions (simple)

If you kept up with adding and subtracting fractions, multiplication and division will be refreshingly simple!

3.1 Multiplying fractions (still simple)

To multiply two (or more) fractions, you simply need to multiply the numerators together, and also to multiply the denominators:

$$\left(\frac{4}{9}\right)\times\left(\frac{3}{6}\right)=\frac{(4\times 3)}{(9\times 6)}=\frac{12}{54}=\frac{2}{9}$$

Note that if both fractions have values smaller than one (as in this case), the answer will be smaller than either of the original fractions.

A fraction raised to a whole-number positive power is a special case of multiplication:

$$\left(\frac{5}{8}\right)^2=\left(\frac{5}{8}\right)\times\left(\frac{5}{8}\right)=\frac{(5\times 5)}{(8\times 8)}=\frac{5^2}{8^2}$$

3.2 Dividing fractions (still pretty simple)

Whilst multiplying fractions looks very much like adding or subtracting, it is not quite so easy to see how you can divide one fraction by another:

$$\left(\frac{5}{8}\right)\div\left(\frac{3}{4}\right)=\left(\frac{5}{8}\right)/\left(\frac{3}{4}\right)$$

This expression is a fraction itself, with the numerator equal to $\frac{5}{8}$ and the denominator equal to $\frac{3}{4}$. By analogy with multiplication, the answer should look like this:

$$\left(\frac{5}{8}\right)\div\left(\frac{3}{4}\right)=\frac{(5\div 3)}{(8\div 4)}$$

However, it is quite likely that either the numerator or denominator will not be a whole number - in this case the numerator is equal to $1\frac{2}{3}$. The way to solve this is to turn the sum into a multiplication, and this is done by inverting the denominator:

$$\left(\frac{5}{8}\right)\div\left(\frac{3}{4}\right)=\left(\frac{5}{8}\right)\times\left(\frac{4}{3}\right)=\frac{(5\times 4)}{(8\times 3)}=\frac{20}{24}=\frac{5}{6}$$

For the purists, we can write this expression as:

$$\left(\frac{5}{8}\right)\div\left(\frac{3}{4}\right)=\left(\frac{5}{8}\right)\times\left(\frac{3}{4}\right)^{-1}=\left(\frac{5}{8}\right)\times\left(\frac{4}{3}\right)$$

4. Decimals (simple)

In section 2, you saw how to convert an improper fraction (one where the numerator is larger than the denominator) into a mixed number. The denominator divides into the numerator to yield a whole number and a fraction, whose numerator is the remainder from the division:

$$\frac{21}{4} = 5\frac{1}{4}$$

If we divide the numerator of the fraction part of the mixed number by the denominator, we convert the fraction into a decimal:

$$5\frac{1}{4} = 5.25$$

The decimal result is usually spoken as 'five point two five' (not 'five point twenty-five').

A decimal number normally comprises a whole number (that can be zero as in the fractions table in section 4.3) and the decimal part (the part of the number that is less than one). The usual separator for these two parts of the number is a period (e.g. '2.345'), although in Continental Europe a comma (e.g. '2,345') is used (for this reason, it is advisable to represent thousands by using a space separator, as in '2 345', rather than a comma). You might occasionally encounter a redundant notation like 3•976, and in the UK a hyphen or dash is used as the separator when writing currency amounts (e.g. £37-58).

4.1 Decimal places (still quite simple)

The decimal part of the number can be thought of as a series of fractions, with the denominator being ten for the first decimal place to the right of the separator, 100 for the second decimal place, 1000 for the third and so on. Take the decimal number 67.329:

	tens	units	separator	tenths	hundreths	thousandths
Decimal place:			.	First	Second	Third
Number:	**6**	**7**	.	**3**	**2**	**9**
Value:	**60**	**7**	.	$\frac{3}{10}$	$\frac{2}{100}$	$\frac{9}{1000}$

Any fractional value smaller than one tenth will have the number zero in the first decimal place, and any fractional value smaller than one hundreth will have zeros in the first and second decimal places. Thus:

$$\frac{1}{25} = 0.04 \quad \text{and} \quad \frac{1}{250} = 0.004$$

Where the decimal result of a calculation does not yield an exact value, the result will normally be expressed as a certain number of decimal places, with the last number being rounded up if the number following it would have been between 5 and 9. So:

Three decimal places	Two decimal places
6.921	**6.92**
6.924	**6.92**
6.925	**6.93**
6.928	**6.93**

Obviously, expressing a value to a specified number of decimal places implies something about the precision of the measurement (see the separate helpsheet on precision if you don't understand this). Say we measure the height of seven seedlings using a 30 cm ruler graduated in centimetres and millimetres. If the seven measurements are 4.7, 6.3, 5.2, 7.1, 6.9, 3.5 and 5.9 cm, we can add these together and divide by 7 to calculate the mean (average) value. A typical result for this on a calculator is 5.6571429 cm. The final '9' represents a length of just 9 nanometres, or nine millionths of a millimetre (you could fit just 300 helium atoms side-by-side into this distance). Clearly, writing the result of the mean to many more decimal places than the original measurements (which were to the nearest millimetre) is absurd, and the average should have been expressed to one decimal place as 5.7 cm (see also helpsheets on measurements and descriptive statistics).

You may be asked to express the result of a calculation to a certain number of decimal places, or you may need to decide for yourself what is appropriate. It is often sensible to state this if you are providing a written answer (e.g. 'result is 3.942 to three decimal places'). The Format>Cells menu in a spreadsheet will allow you to set the number of decimal places that are displayed (the spreadsheet will round the result for you).

4.2 Significant figures (still quite simple)

You may also come across the related concept of 'significant figures'. This is based on the same ideas about precision, but extends to the entire decimal number rather than just the decimal part. Let's look at a real-life example. Looking on Wikipedia, I find that the Earth's orbital period (or 'siderial' year) is 365.25636042 days (see http://en.wikipedia.org/wiki/Sidereal_year). This value means that the Earth does not go once around the Sun in an exact number of days - the journey takes about a quarter of a day longer than 365 'whole' days. To keep our calendars in pace with this, we have leap years every four years, where we pretend that it takes 366 years to orbit the Sun once. However, if we did this every four years we would gradually creep ahead, which is why we miss out leap years occasionally (eg at the start of a century).

So the 'real' value is 365.25636042 years (where that last '2' is 0.00000002 of a day or about one millisecond). In practical terms, we take the value of a year to be 365 days, which is using the first three digits of the value, which we also call 'expressing the value to three significant figures'. If we want to include the adjustment for leap years in most four year intervals, we would describe the length of a year as 365.26 years. This value has five significant figures, and better describes the length of the year. That there is a 6 rather than a 5 or a 7 in the second decimal place implies that we can describe the length of the year with a certainty or precision of one part in one hundred thousand - equivalent to a clock that gains or loses five minutes in the space of a year. If we include eight significant figures the value is 365.25636, and the value in the last decimal place implies a precision of one part in one hundred million - a clock that gains or loses about a third of a second over one year.

The number of decimal places and the number of significant figures are clearly related, and both are fundamentally concerned with the precision of the value. The number of significant figures takes into account the size of the whole-number part, as these examples show:

Value	Number of decimal places	Number of significant figures
1.4673	**4**	**5**
613.4673	**4**	**7**
613.47	**2**	**5**
37613.47	**2**	**7**
37613.4700	**4***	**9***

* Note that the value in the last row is numerically identical to that in the row above, but is expressed to an additional two decimal places and two significant figures.

4.3 Common fractions expressed as decimals

Several common fractions have exact and simple decimal equivalents, whilst others are either complicated or do not have exact decimal solutions. For instance, $\frac{1}{3}$ has the decimal equivalent 0.3333333333...... This is denoted '0.3 recurring', and is written as $0.333^{\cdot}$.

$\frac{1}{25}$	$\frac{1}{20}$	$\frac{1}{18}$	$\frac{1}{16}$	$\frac{1}{15}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$
0.040	**0.050**	**$0.055^{\cdot}$**	**0.0625**	**$0.066^{\cdot}$**	**$0.083^{\cdot}$**	**$0.111^{\cdot}$**	**0.125**	***0.143***	**$0.166^{\cdot}$**
$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{7}{8}$	$\frac{3}{16}$	$\frac{5}{16}$
$0.333^{\cdot}$	**$0.666^{\cdot}$**	**0.250**	**0.500**	**0.750**	**0.375**	**0.625**	**0.875**	**0.1875**	**0.3125**
$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{5}{10}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	
0.100	**0.200**	**0.300**	**0.400**	**0.500**	**0.600**	**0.700**	**0.800**	**0.900**	

All decimal values are shown to three decimal places (ie 0.NNN) except where there is an exact solution with four decimal places. The superscripted dot indicates a recurrent solution, and the single italicized value is an approximation rounded to three decimal places.

5. Ratios, proportions and percentages (reasonably simple)

In section 4.2, you will have seen the concept of precision expressed by analogy with a clock that gains or loses a certain amount within a period of time. Taking an extreme example of a clock that gains a minute a day, we can say that the clock gains a minute every 1440 minutes. This can be written as a 'ratio', usually in the form '1:1440', '1/1440' or 'one part in 1440'. Note that the change (the gain) and the base measurement (elapsed time) are both expressed in the same unit of measurement, so that the ratio has no units and is said to be 'dimensionless'. This is not always the case, and if units for the two quantities differ, the unit(s) should be indicated.

You may also meet the term ratio used erroneously to describe the recipe for a mixture. For instance, cement mortar for bricklaying is made by mixing one part of Portland cement with five parts of sand. This is written as 1:5 (e.g. http://www.diydata.com/materials/cement/cement.php), but this is not the same as the ratio described in the previous paragraph.

If you take the ratio 1:1440 described earlier, and divide the gain by the elapsed time, the result (0.0007) expresses the gain as a 'proportion' of the elapsed time. This is another way of writing the same relationship, and again it is dimensionless in this case. The proportion can be used to calculate how much the clock gains over other periods, for instance over a week (10 080 minutes) it will gain 0.0007 × 10 080 = 7 minutes.

A ratio or proportion is normally a small thing expressed as part of a larger thing, but it doesn't have to be so. A ratio of 2:1 or a proportion of 2 both express a change that is twice the value of the base measurement. However, in most cases, proportions tend to be less than one, and are often quite small numbers. For instance, if a bank account pays £5 interest for each £100 invested over a period of one year, the interest expressed as a proportion is 0.05 per year (note that this proportion has a unit). For a variety of reasons, proportions like this are often multiplied by 100 to yield a quantity called a 'percentage' - in this example the interest would be advertised by the bank as '5% per year'. Spreadsheets and most calculators will allow you to work directly with percentages, but be careful that you understand fully how you are using them. If you are in any doubt, it is better to use proportions.

$$percentage = proportion \times 100 \quad \text{and} \quad proportion = \frac{percentage}{100}$$

Another way that you will need to use percentages or proportions is to work out the final value or cost of something subject to increase. What does a 3.25% pay award mean to someone earning £19 565 per year? What will be the price of a power drill marked at £34-99 plus VAT? For the VAT example, value-added tax is charged at 17.5% of the purchase price, so that:

price including VAT = price excluding VAT + 17.5% of the ex-VAT price

Many people will work out the VAT payable and then add this to the price. However, if you understand the relationship between percentages and proportions, you can do the calculation in one go. Remember that a percentage of 17.5% is equal to a proportion of 0.175. So the equation above can be re-written as:

price including VAT = ex-VAT price + 0.175 × ex-VAT price = 1.175 × ex-VAT price

so that:

£34-99 × 1.175 = £41-11

Returning to the pay increase, the new salary will be £19 565 × 1.0325 = £20 201.

The way that things like interest rates are reported in the media can cause confusion. An increase in the mortgage interest rate of 'half a percentage point' sounds fairly benign. However, if the current rate is 3.75%, the rate has now increased to 4.25% - this represents an increase of 13% over the previous rate, which would be reflected in monthly payments.

All ratios, proportions and percentages are interconvertable:

Ratio	Proportion	Percentage
$1:R$	$\frac{1}{R}$	$\left(\frac{1}{R}\right)\times 100$
1:20	**0.2**	**20%**
1:200	**0.02**	**2%**
1:2000	**0.002**	**0.2%**
2:1	**2**	**200%**