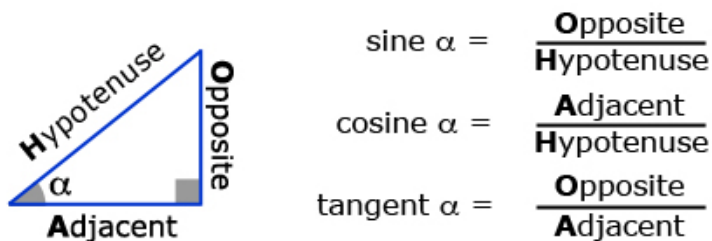


Introduction to functions and models: UNDERSTANDING AND USING TRIGONOMETRIC FUNCTIONS

1. Revision of the three basic trigonometric functions (simple)

The three basic trigonometric functions are defined for a right-angled triangle as ratios between pairs of sides as shown in the diagram.



You will notice that for the third angle of the triangle, which we can designate β :

$$\begin{aligned} \beta &= 90 - \alpha \\ \text{sine } \beta &= \text{cosine } \alpha \\ \text{cosine } \beta &= \text{sine } \alpha \\ \text{tangent } \beta &= 1/\text{tangent } \alpha \end{aligned}$$

If you need to find further information on these functions, you should consult the helpsheet **Basic numerical skills: TRIANGLES AND TRIGONOMETRIC FUNCTIONS**.

2. Degrees and radians (simple)

We are used to measuring angles in degrees, where a right angle is 90° , the angle subtended by a straight line is 180° and a full revolution of a circle is 360° . We can also measure angles using a unit called a 'radian', which is based on the relationship between the circumference of a circle and its radius. Imagine an arc of a circle whose length is equal to the circle's radius. If you draw the radii at each end of the arc, the angle that they make at the centre of the circle is 1 radian. A full revolution of a circle is expressed as 2π radians, so:

$$\begin{aligned} 90^\circ &= 0.5\pi \text{ radians} \\ 180^\circ &= 1\pi \text{ radians} \\ 270^\circ &= 1.5\pi \text{ radians} \\ 360^\circ &= 2\pi \text{ radians} \end{aligned}$$

Generally, if we have an angle measured in degrees, it can be converted to radians using the following formula:

$$\alpha^\circ = (\alpha \cdot \pi/180) \text{ radians}$$

and conversely:

$$\alpha \text{ radians} = (\alpha \cdot 180/\pi)^\circ$$

Note spreadsheets and many calculators use radians as the default measurement when calculating trigonometric functions. Thus $\text{SIN}(2)$ returns the result 0.91, which is the sine of 2 radians, whereas the sine of 2° is 0.035.

3. Going round in circles (intermediate)

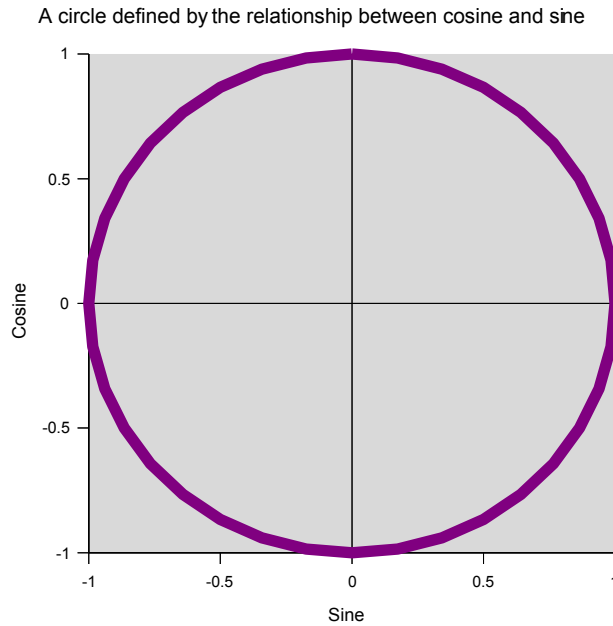
The radian unit links the angle and its trigonometric functions directly to a circle. If you have a

point moving at a constant distance from a second, fixed point, the path of the first point describes a circle. We can describe the location of the point in terms of the angle that the line joining the two points (the radius) makes with the vertical. For a distance from the centre of r , the x- and y-coordinates of the moving point are given by:

$$x = r \sin\alpha$$

$$y = r \cos\alpha$$

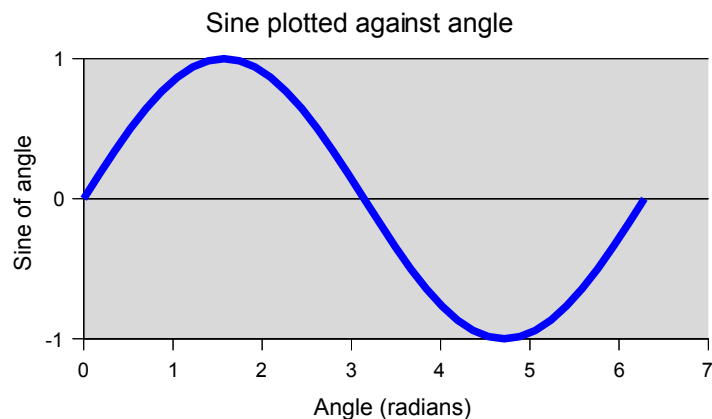
where the angle α is measured clockwise from the vertical (ie the y-axis). So if we plot the cosine of an angle against its sine, we draw a perfect circle with a radius of one unit ($r = 1$):



You can see that for angles up to 0.5π radians (90°), both the sine and cosine have positive values. Between 0.5π radians (90°) and 1π radians (180°), the sine is still positive but the cosine is negative. From 1π radians (180°) to 1.5π radians (270°), both are negative, and then from 1.5π radians (270°) to 2π radians (360°) the sine remains negative but the cosine is positive.

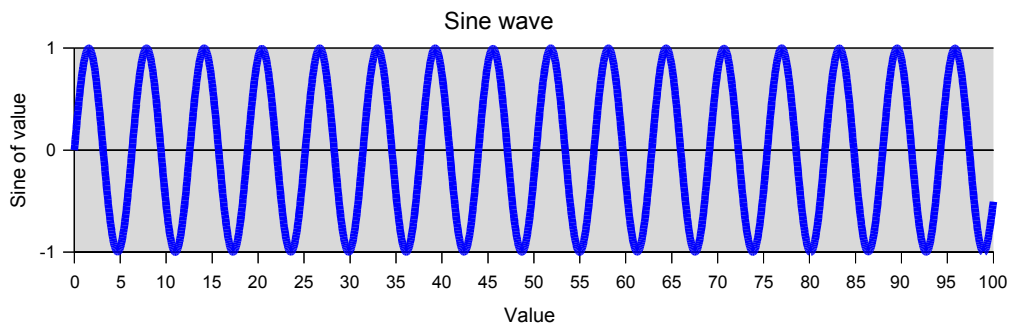
4. Making (sine) waves (advanced)

The relationship between the sine of an angle and the value of the angle describes a cycle or wave that looks like:



The sine wave starts with a value of zero, increases to 1 at 0.5π radians, is zero again at 1π radians, is -1 at 1.5π radians and returns to zero at 2π radians. As you will appreciate from section 3, the cosine wave is a similar shape, but starts with a value of 1 (ie the value of sine at 0.5π radians).

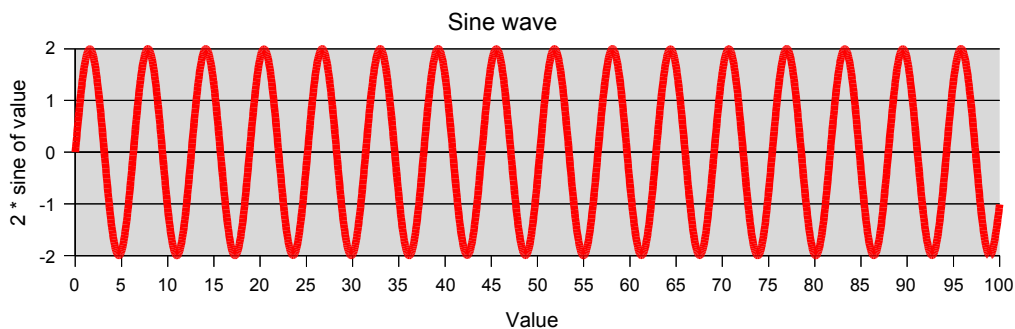
The sine wave doesn't have to stop at 2π radians. Although it is hard to conceive of an angle of, say, 4π radians (720°), it is perfectly feasible to have the sine of 4π radians. You shouldn't be surprised to find that the sine of 4π radians is also zero, because the sine wave carries on repeating itself. In fact, you can calculate the sine of any real number (positive or negative) - a sine wave calculated for values between zero and 100 looks like this:



Notice that the wave still has a range from 1 to -1 – its amplitude is 2. By adding extra variables to the sine wave, it can be changed to make it have whatever values you want. You can change the amplitude by adding a multiplier to the sine wave, for instance:

$$fx = 2 \sin x$$

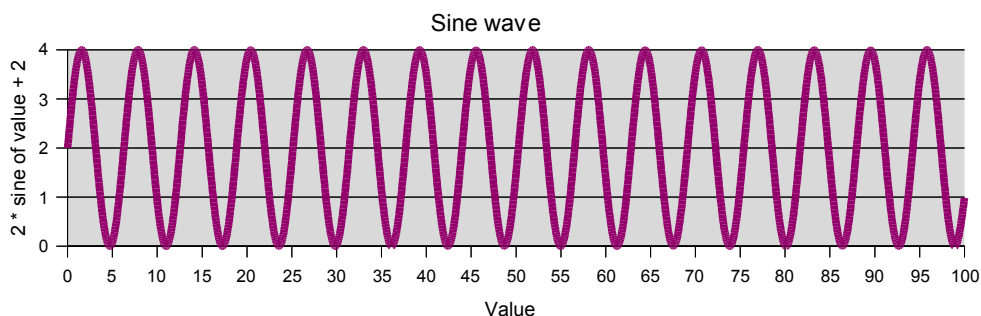
which has an amplitude of 4:



This transformed sine wave still has an average value of zero. If we add a constant to it, we leave the amplitude unchanged but change the maximum and minimum values, for instance:

$$fx = 2 \sin x + 2$$

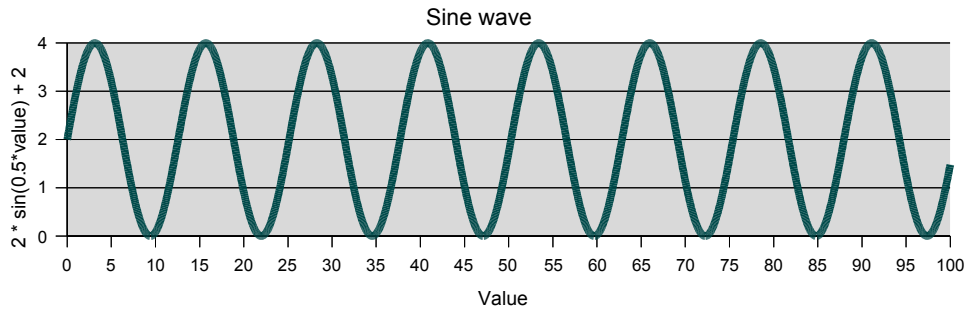
will range from zero to +4 instead of -2 to +2:



The frequency of the sine wave is unchanged from the original plot – it goes through a complete cycle in 2π (about 6.3). We can transform the sine wave further to change the frequency, for example:

$$fx = 2 \sin(0.5x) + 2$$

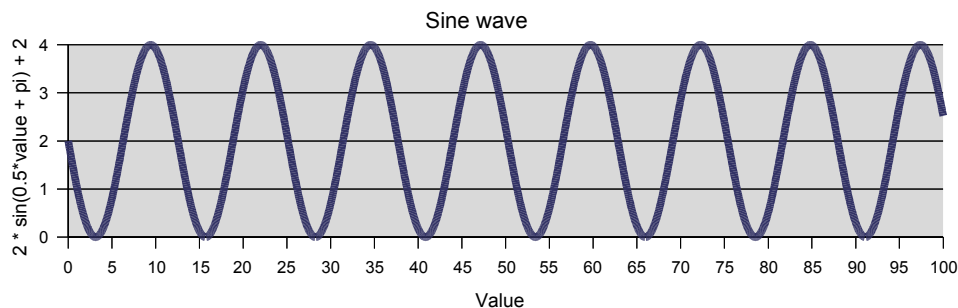
will have a frequency that is half of that of our original wave – the distance between adjacent peaks has been doubled:



Finally, we can shift our wave sideways so that it doesn't start with a value of half of the overall amplitude. By adding an offset to the **argument** for the sine (the bit in brackets), we can shift the curve sideways by a predictable amount:

$$fx = 2 \sin(0.5x + \pi) + 2$$

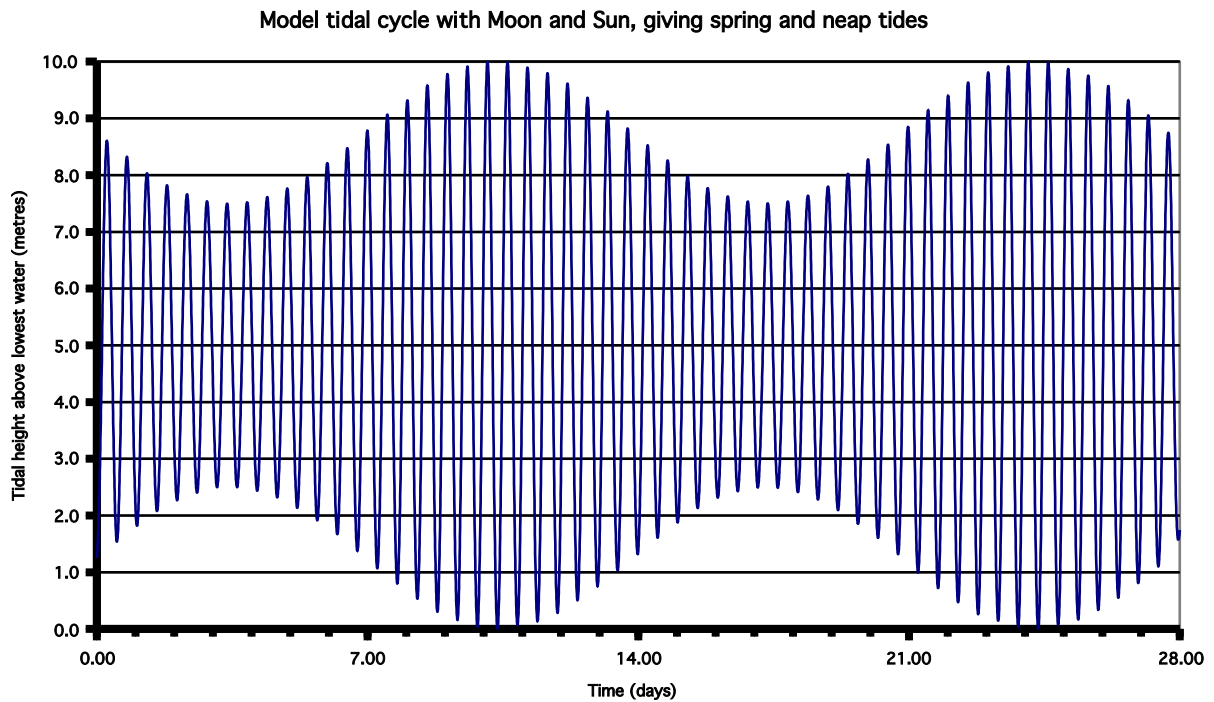
Because a complete cycle of the sine wave occupies 2π , adding an offset of 1π shifts the curve by half a cycle, which will reverse the phase (turn peaks into troughs), as:



If you are going to use this technique to produce a regular cycle, you need to be very clear what each of the bits of the expression does, and be careful where you place your brackets. It is very easy to implement this using a spreadsheet, which is where this series of diagrams originated.

5. Using transformed sine waves to model real systems (advanced)

We have demonstrated that a sine wave can be 'customized' to simulate any regular cycle, such as a diurnal or seasonal cycle. You can even combine sine waves, for instance to produce a very crude representation of a monthly tidal cycle:



The spreadsheet-based model in the diagram combines two sine waves. The first simulates the regular cycle of high and low tides (with a period of about 12.4 hours), whilst the second is a longer-period cycle that represents – very crudely – the spring-neap cycle with a period of about 14 days. Note that the curve has not been smoothed out, so it appears jagged as straight lines join data points that are two hours apart. The two cycles were constructed using the same approach used in section 4 to transform a single sine wave.