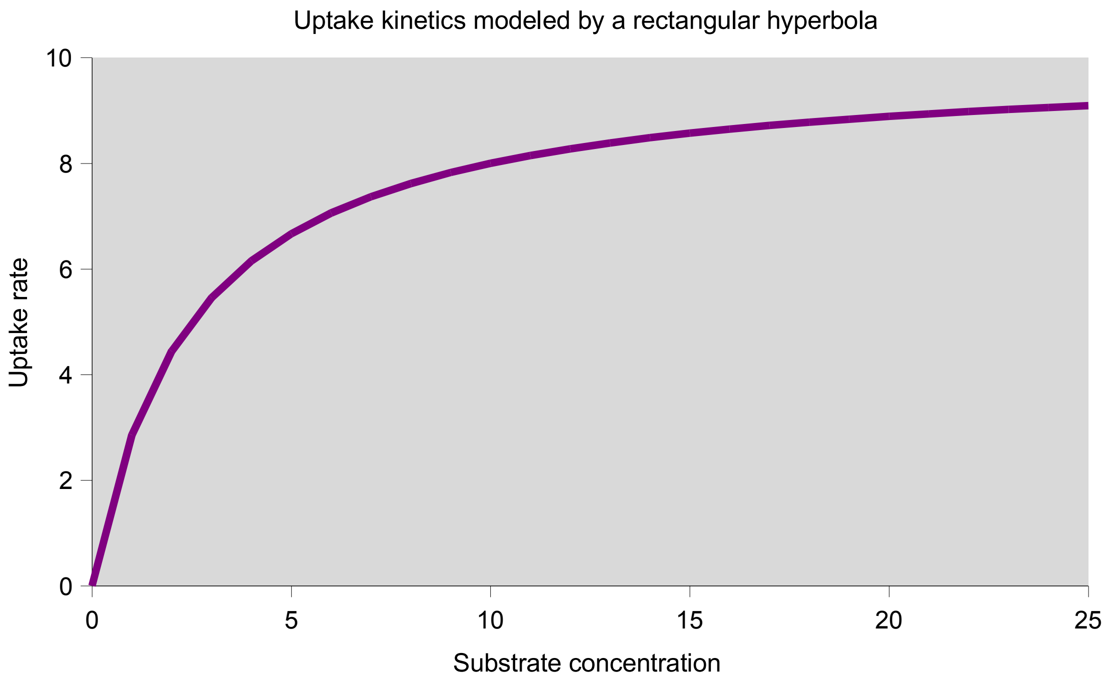
Introduction to functions and models: MODELS INVOLVING HYPERBOLIC FUNCTIONS

1. Introduction

Hyperbolic models are used to describe several types of relationships. What they have in common is a specified maximum value of the dependent variable, and a description of when the maximum value is attained in respect of the independent variable. There are several different types of hyperbolic functions. In this worksheet we will explore some commonly-used models in biosciences that use hyperbolic functions.

2. The Michaelis-Menten model of substrate uptake – a simple rectangular hyperbola (rating intermediate)

Studies of the growth rates of microbes under controlled conditions commonly examine the relationship between some measure of growth rate, such as the incorporation of a radio-labelled substrate, and the substrate concentration. An example might be the uptake rate of bacteria grown at different concentrations of glucose, measured by the incorporation of ^{14}C-labelled glucose. Such a relationship is very rarely linear, because microorganisms absorb dissolved materials through their cell membranes and either the cell surface area or the number of transport sites in the cell membrane place an upper limit on the amount of substrate that a cell can absorb over a given time. Because of this, the relationship typically looks like this:



At low substrate concentrations, there is a roughly linear relationship between the uptake rate and the substrate concentration. This means that if the concentration is doubled, the uptake rate will also double. But at higher concentrations, the cell cannot continue absorbing further substrate at the same rate. Here, the rate at which uptake rate increases declines gradually as the uptake rate approaches the maximum that the cell can sustain. The curve in the figure was produced using a rectangular hyperbolic model, which is often called Michaelis-Menten kinetics in this context. The model takes this form:

$$v = v_{max} \,.\, (S/[S+K_m])$$

In this model:

v is the uptake rate at a substrate concentration of S
v_{max} is the maximum uptake rate
K_m is a constant, which is the substrate concentration at which the uptake rate is half of v_{max}.

If you look at the part of the rectangular hyperbola that is enclosed by brackets, $(S/[S+K_m])$, you can see that as the substrate concentration increases, S becomes much larger than K_m. This means that the value of $(S/[S+K_m])$ tends towards one, and v tends towards v_{max}.

We can show that K_m is the substrate concentration where v is half of v_{max} by considering the case where S is equal to K_m:

$$S = K_m$$

$$(S/[S+K_m]) = (K_m/[K_m+K_m]) = 0.5$$

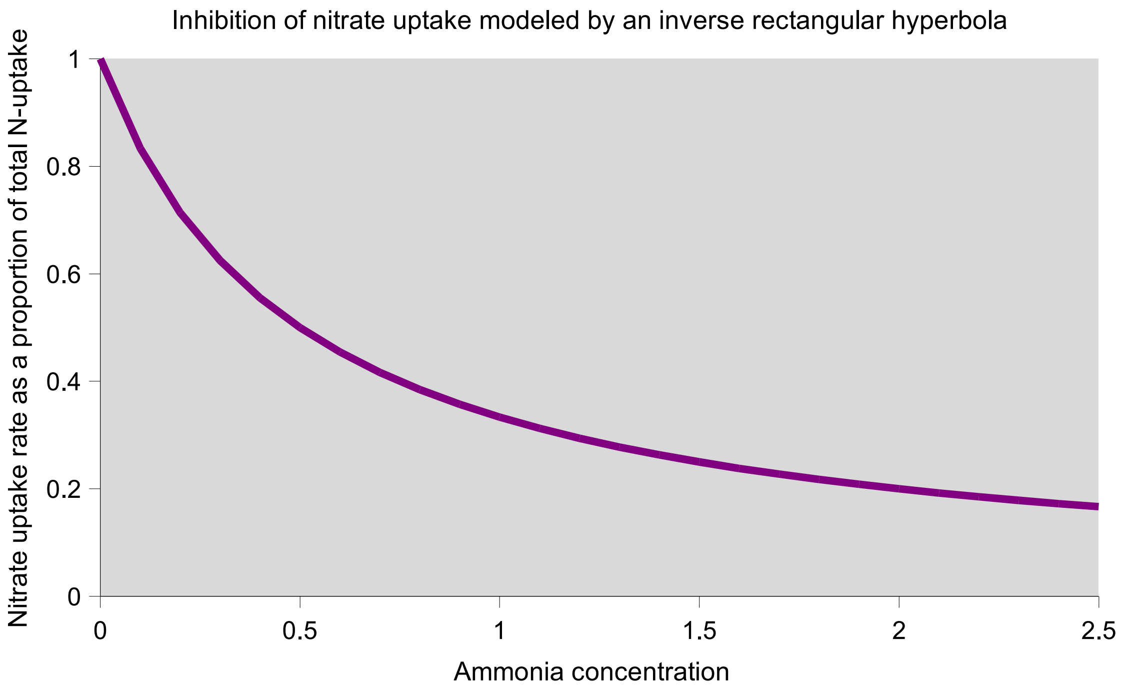
$$v = v_{max} \,.\, 0.5$$

If the value of K_m is small, the initial slope of the curve is steep and the uptake rate soon approaches the value of v_{max}. Conversely, if K_m is large, the initial slope is small and uptake rates approaching v_{max} are only attained at very high substrate concentrations. In the model illustrated above, the value of v_{max} is 10, whilst K_m is 2.5. Note that even at the highest substrate concentration, which is ten times the value of K_m, v is clearly less than v_{max}.

3. Modeling uptake inhibition using an inverse hyperbola (rating intermediate)

In the previous example, the uptake rate started at zero at a substrate concentration of zero, and increased towards a maximum value as the substrate concentration increased. Models based on hyperbolic functions can also be used to describe a different situation, where the uptake rate starts at a maximum value and decreases as the substrate concentration increases.

An example is provided by marine microalgae (phytoplankton). At some times of the year, they have access to two main sources of nitrogen dissolved in the seawater. Nitrate is abundant, but the microalgae need to expend energy to transport nitrate into the cell. By contrast, dissolved ammonia is normally scarce (perhaps only 10% of the nitrate concentration) but is 'free' in that ammonia can diffuse through the cell membrane without any energy cost. If ammonia concentration in seawater increases, microalgae switch from using nitrate to using ammonia. A model of the uptake rate of nitrate (as a proportion of total nitrogen uptake) in relation to ammonia concentration would look like this:



Here, the uptake rate starts at its maximum value, which is 1 as all of the nitrogen uptake is in the form of nitrate. Nitrate uptake rate declines steeply initially, and then the curve gradually levels out. The curve in the figure was produced from an inverse rectangular hyperbolic model. The model takes this form:

$$v = v_{max} \,.\, 1 - (S/[S+K_m])$$

We have used the same notation as in the Michaelis-Menten model, so that:

v is the uptake rate at a substrate concentration of S
v_{max} is the maximum uptake rate (when the concentration of the inhibiting substrate, ammonia, is zero)
K_m is a constant, which is the ammonia concentration at which the nitrate uptake rate is half of v_{max}

Look at the expression $1 - (S/[S+K_m])$ in the inverse hyperbola equation. You should see that for an ammonia concentration (S) of zero, $(S/[S+K_m])$ also has a value of zero. One minus zero is one, so v equals v_{max}. As the substrate concentration increases significantly above K_m, S becomes much larger than K_m, and the value of $(S/[S+K_m])$ tends towards one, and v tends towards zero. However, v will only come close zero when S is very much larger than K_m.

If the value of K_m is small, the initial (negative) slope of the curve is steep and the uptake rate soon decreases well below the value of v_{max}. Conversely, if K_m is large, the initial slope is small and uptake rate declines more slowly. In the model illustrated above, the value of v_{max} is 1, whilst K_m is 0.5.

4. Photosynthesis-light curves: another model in the hyperbola family (rating advanced)

In previous examples, we have considered the use of a simple hyperbolic function to model the relationship between substrate uptake rate and substrate concentration. The examples in previous section used dissolved chemicals as substrates. Here, we are concerned with photosynthesis, where the substrate is light energy rather dissolved chemical nutrients, but the principles are identical and the shape of the curve is very similar. There is a wide variety of models of the relationship between photosynthetic rate and light (including variants of the rectangular hyperbola used earlier). Here, we are using a model that behaves in a similar way but is formulated in a different way.

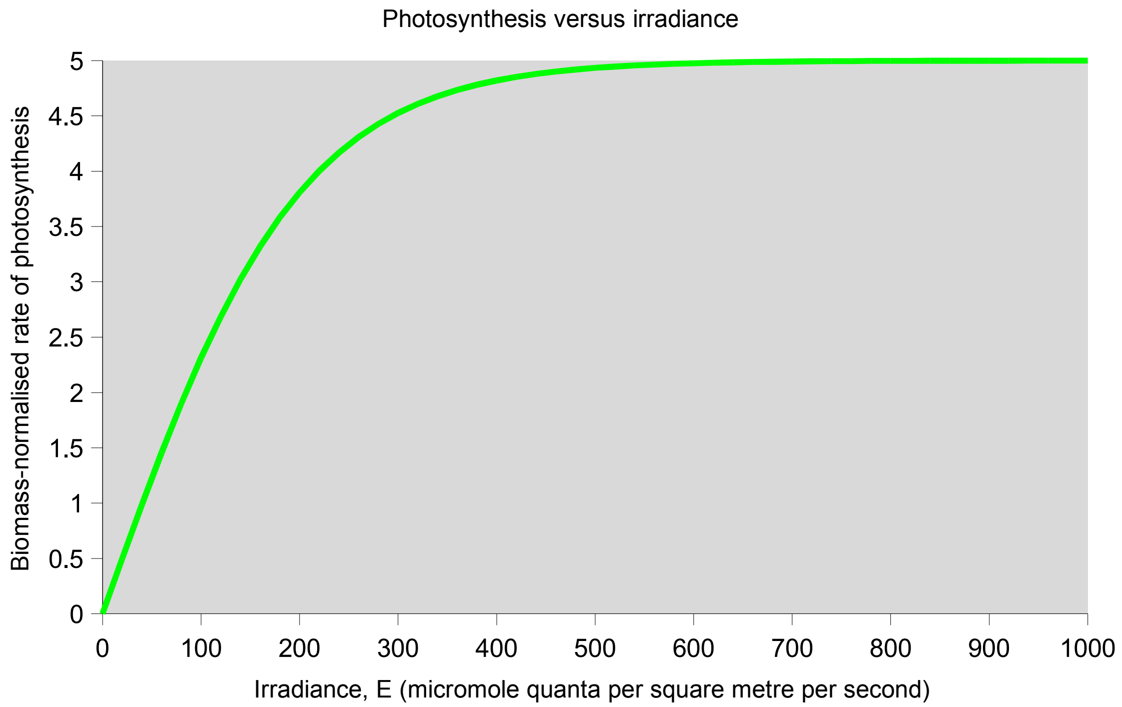
All of the models described here are approximations – they are not necessarily based on realistic physiology although the parameters (constants) of the model can be interpreted in terms of physiological processes. This model of the relationship between photosynthetic rate and irradiance (light level) was selected by Trevor Platt and colleagues as being a good representation under a wide range of conditions, and also easy to fit to real data.

The model uses a hyperbolic trigonometric function, the hyperbolic tangent or 'tanh' (pronounced 'tansh'). This function is usually available in spreadsheets, so the model can be built easily in a spreadsheet. The model is written as:

$$P = P_{\max} \,.\, \tanh(\alpha . E / P_{\max})$$

P is the rate of photosynthesis at an irradiance of E
P_{max} is the maximum photosynthetic rate
α is a constant corresponding to the initial slope of the curve, so has units of P/E (making the argument of the tanh dimensionless)

As E increases, the value of $\tanh(\alpha . E / P_{\max})$ tends towards 1 and P tends towards $P_{\max}$. The curve produced by the model looks like this:



Unlike the rectangular hyperbolae used in previous sections, the maximum value $P_{\max}$ is achieved within the range of irradiances modelled and so has some physiological relevance.

This model does not include the commonly observed inhibition of photosynthesis at very high irradiances. It can be modified to include an inhibition term, or other photosynthesis-irradiance models can be used.